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Weyl's theorem for upper triangular operator matrices

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Abstract

Let $\sigma_{ab}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not an upper semi-Fredholm operator with finite ascent}\}$ be the Browder essential approximate point spectrum of $T \in B(H)$ and let $\sigma_d(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not surjective}\}$ be the surjective spectrum of T . In this paper it is shown that if $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is a 2×2 upper triangular operator matrix acting on the Hilbert space $H \oplus K$, then the passage from $\sigma_{ab}(A) \cup \sigma_{ab}(B)$ to $\sigma_{ab}(M_C)$ is accomplished by removing certain open subsets of $\sigma_d(A) \cap \sigma_{ab}(B)$ from the former, that is, there is equality

$$\sigma_{ab}(A) \cup \sigma_{ab}(B) = \sigma_{ab}(M_C) \cup \mathcal{G},$$

where \mathcal{G} is the union of certain of the holes in $\sigma_{ab}(M_C)$ which happen to be subsets of $\sigma_d(A) \cap \sigma_{ab}(B)$. Weyl's theorem and Browder's theorem are liable to fail for 2×2 operator matrices. In this paper, it also explores how Weyl's theorem, Browder's theorem, a-Weyl's theorem and a-Browder's theorem survive for 2×2 upper triangular operator matrices on the Hilbert space. © 2005 Elsevier Inc. All rights reserved.

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1. Introduction

The study of upper triangular operator matrices arises naturally from the following fact: if T is a Hilbert space operator and M is an invariant subspace for T , then T has the following 2×2 upper triangular operator matrix representation:

$$T = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} : M \oplus M^\perp \longrightarrow M \oplus M^\perp,$$

and one way to study operators is to see them as entries of simpler operators. The upper triangular operator matrices have been studied by many authors. This paper is concerned with the Browder essential approximate point spectrum of 2×2 upper triangular operator matrices. We also study Weyl's theorem and a-Weyl's theorem for 2×2 upper triangular operator matrices.

Throughout this paper, let H and K be Hilbert spaces, let $B(H, K)$ denote the set of bounded linear operators from H to K , and abbreviate $B(H, H)$ to $B(H)$. If $A \in B(H)$, write $N(A)$ for the null space of A and $R(A)$ for the range of A . For $A \in B(H)$, if $R(A)$ is closed and $\dim N(A) < \infty$, we call A upper semi-Fredholm operator and if $\dim H/R(A) < \infty$, then A is called lower semi-Fredholm operator. Let $\Phi_+(H)$ ($\Phi_-(H)$) denote the set of all upper (lower) semi-Fredholm operators on H . An operator A is called Fredholm operator if $A \in \Phi_+(H) \cap \Phi_-(H)$. If A is semi-Fredholm operator and let $n(A) = \dim N(A)$ and $d(A) = \dim H/R(A)$, then we define the index of A by $\text{ind}(A) = n(A) - d(A)$. An operator A is called Weyl operator if it is a Fredholm operator of index zero, and is called Browder if it is Fredholm "of finite ascent and descent". We write $\alpha(A)$ and $\beta(A)$ for the ascent and the descent for $A \in B(H)$ respectively. If $A \in B(H)$, write $\sigma(A)$ for the spectrum of A ; $\sigma_a(A)$ for the approximate point spectrum of A ; $\pi_{00}(A)$ for the isolated points of $\sigma(A)$ which are eigenvalues of finite multiplicity; $\pi_{00}^a(A)$ for the isolated points of $\sigma_a(A)$ which are eigenvalues of finite multiplicity. Let $\rho(A) = \mathbb{C} \setminus \sigma(A)$ and $\rho_a(A) = \mathbb{C} \setminus \sigma_a(A)$. The essential spectrum $\sigma_e(A)$, the Weyl spectrum $\sigma_w(A)$, the Browder spectrum $\sigma_b(A)$ of A are defined by: $\sigma_e(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Fredholm}\}$; $\sigma_w(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Weyl}\}$; $\sigma_b(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Browder}\}$. We say that Weyl's theorem holds for $A \in B(H)$ if

$$\sigma(A) \setminus \sigma_w(A) = \pi_{00}(A)$$

and Browder's theorem holds for A [5] if

$$\sigma_w(A) = \sigma_b(A).$$

Clearly, Weyl's theorem implies Browder's theorem.

Let $\Phi_+^-(H)$ be the class of all $A \in \Phi_+(H)$ with $\text{ind}(A) \leq 0$, and for any $A \in B(X)$, let

$$\sigma_{SF_+}(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not in } \Phi_+(X)\}$$

and

$$\sigma_{ea}(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not in } \Phi_+^-(X)\}.$$

$\sigma_{ea}(A)$ is called the essential approximate point spectrum of A and $\sigma_{ab}(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not an upper semi-Fredholm operator with finite ascent}\}$ is called Browder essential approximate point spectrum of A .

We say that a-Weyl's theorem holds for A if there is equality

$$\sigma_a(A) \setminus \sigma_{ea}(A) = \pi_{00}^a(A)$$

and that a-Browder's theorem holds for A if there is equality

$$\sigma_{ea}(A) = \sigma_{ab}(A).$$

It is known [1,2] that if $A \in B(H)$, then we have

$$\text{a-Weyl's theorem} \implies \text{Weyl's theorem} \implies \text{Browder's theorem};$$

$$\text{a-Weyl's theorem} \implies \text{a-Browder's theorem} \implies \text{Browder's theorem}.$$

Recall that an operator $A \in B(H)$ is said to be bounded below if there is $k > 0$ for which $\|x\| \leq k\|Ax\|$ for each $x \in H$. A is bounded below if and only if 0 is not in $\sigma_a(A)$. If \mathcal{G} is a compact subset of \mathbb{C} , we write $\text{int } \mathcal{G}$ for the interior points of \mathcal{G} ; $\text{iso } \mathcal{G}$ for the isolated points of \mathcal{G} ; $\text{acc } \mathcal{G}$ for the accumulation points of \mathcal{G} ; $\partial \mathcal{G}$ for the topological boundary of \mathcal{G} . $A \in B(H)$ is called approximate-isoloid (abbrev. a-isoloid) if every isolated point of $\sigma_a(A)$ is an eigenvalue of A and A is called isoloid if every isolated point of $\sigma(A)$ is an eigenvalue of A . When $A \in B(H)$ and $B \in B(K)$ are given, we denote by M_C an operator acting on $H \oplus K$ of the form

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

where $C \in B(K, H)$. In Section 2, we characterize the Browder essential approximate point spectrum of M_C . In Section 3, we explore how Weyl's theorem, Browder's theorem, a-Weyl's theorem and a-Browder's theorem survive for 2×2 upper triangular operator matrix M_C .

2. The Browder essential approximate point spectrum for upper triangular operator matrices

In [4], it is shown that the passage from $\sigma(A) \cup \sigma(B)$ to $\sigma(M_C)$ is accomplished by removing certain open subsets of $\sigma(A) \cap \sigma(B)$ from the former, that is, there is equality

$$\sigma(A) \cup \sigma(B) = \sigma(M_C) \cup W,$$

where W is the union of certain of the holes in $\sigma(M_C)$ which happen to be subsets of $\sigma(A) \cap \sigma(B)$. However we need not expect the case for the Browder essential approximate point spectrum. The passage from $\sigma_{ab}(A) \cup \sigma_{ab}(B)$ to $\sigma_{ab}(M_C)$ is more delicate.

Suppose A is an upper semi-Fredholm operator, using the perturbation theorem of semi-Fredholm operator [7, Theorem 5.31], $A - \lambda I$ is upper semi-Fredholm and

$n(A - \lambda I)$, $d(A - \lambda I)$ are constant for sufficiently small $|\lambda| > 0$. In this case, we also have that $N(A - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(A - \lambda I)^n]$ if $|\lambda| > 0$ is sufficiently small. In fact, If A is semi-Fredholm, let $M = \bigcap_{n=1}^{\infty} R(A^n)$ and let $A_1 = A|_M$, then M is closed and A_1 is surjective. By perturbation theorem of semi-Fredholm operator, $A_1 - \lambda I$ is surjective if $|\lambda| > 0$ is sufficiently small [7, Theorem 5.22], which means $(A_1 - \lambda I)^n M = M$ for any n . It induces that $M \subseteq \bigcap_{n=1}^{\infty} R[(A_1 - \lambda I)^n] \subseteq \bigcap_{n=1}^{\infty} R[(A - \lambda I)^n]$. Since $N(A - \lambda I) \subseteq M$ for any $\lambda \neq 0$, it follows that $N(A - \lambda I) \subseteq M \subseteq \bigcap_{n=1}^{\infty} R[(A - \lambda I)^n]$ if $|\lambda| > 0$ is sufficiently small.

For $A \in B(H)$, let $\sigma_{SF-}(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not lower semi-Fredholm operator}\}$.

Lemma 2.1. *For a given pair (A, B) of operators, there is equality, for every $C \in B(K, H)$,*

$$\eta(\sigma_{\bar{w}}(M_C)) = \eta(\sigma_{\bar{w}}(A) \cup \sigma_{\bar{w}}(B)),$$

where $\sigma_{\bar{w}} \in \{\sigma_b, \sigma_{ea}, \sigma_{ab}, \sigma_{SF+}, \sigma_e, \sigma_{SF-}\}$ and $\eta(F)$ denote the “polynomially convex hull” of the compact set $F \subseteq \mathbb{C}$.

Proof. We only prove that for every $C \in B(K, H)$,

$$\eta(\sigma_{ab}(M_C)) = \eta(\sigma_{ab}(A) \cup \sigma_{ab}(B)),$$

the other cases have the same proof. First it will prove that for every $T \in B(H)$, $\eta(\sigma_{ea}(T)) = \eta(\sigma_w(T))$. In fact, since $\sigma_{ea}(T) \subseteq \sigma_w(T)$, we only need to prove that $\partial\sigma_w(T) \subseteq \sigma_{ea}(T)$. If there is $\lambda_0 \in \partial\sigma_w(T) \setminus \sigma_{ea}(T)$, that is $T - \lambda_0 I \in \Phi_+^-(H)$. By perturbation theory of upper semi-Fredholm operator, there exists $\epsilon > 0$ such that $T - \lambda I \in \Phi_+^-(H)$ and $\text{ind}(T - \lambda I) = \text{ind}(T - \lambda_0 I)$ if $0 < |\lambda - \lambda_0| < \epsilon$ [7, Theorem 5.22]. Since $\lambda_0 \in \partial\sigma_w(T)$, there exists λ_1 such that $0 < |\lambda_1 - \lambda_0| < \epsilon$ and $T - \lambda_1 I$ is Weyl. Then $\text{ind}(T - \lambda_0 I) = 0$, which means that $T - \lambda_0 I$ is Weyl. It is in contradiction to the fact that $\lambda_0 \in \sigma_w(T)$. Then $\partial\sigma_w(T) \subseteq \sigma_{ea}(T)$ and hence $\eta(\sigma_{ea}(T)) = \eta(\sigma_w(T))$. Similarly, for every $T_1 \in B(H)$ and $T_2 \in B(K)$, $\eta(\sigma_{ea}(T_1) \cup \sigma_{ea}(T_2)) = \eta(\sigma_w(T_1) \cup \sigma_w(T_2))$. Second, we will prove that $\eta(\sigma_{ab}(T)) = \eta(\sigma_{ea}(T))$ and $\eta(\sigma_{ea}(T_1) \cup \sigma_{ea}(T_2)) = \eta(\sigma_{ab}(T_1) \cup \sigma_{ab}(T_2))$. Clearly, $\sigma_{ea}(T) \subseteq \sigma_{ab}(T)$. We need to prove $\partial\sigma_{ab}(T) \subseteq \sigma_{ea}(T)$. If $\lambda_0 \in \partial\sigma_{ab}(T) \setminus \sigma_{ea}(T)$, then there exists $\epsilon > 0$ such that $T - \lambda I \in \Phi_+^-(H)$, $N(T - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda I)^n]$ and $n(T - \lambda I)$ is constant if $0 < |\lambda - \lambda_0| < \epsilon$. Since $\lambda_0 \in \partial\sigma_{ab}(T)$, there exists λ_1 such that $0 < |\lambda_1 - \lambda_0| < \epsilon$ and $T - \lambda_1 I \in \Phi_+^-(H)$ with finite ascent. Then $N(T - \lambda_1 I) = N(T - \lambda_1 I) \cap \bigcap_{n=1}^{\infty} R[(T - \lambda_1 I)^n] = \{0\}$ [12, Lemma 3.4], which means that $n(T - \lambda_1 I) = 0$. Thus $T - \lambda_1 I$ is bounded below if $0 < |\lambda - \lambda_0| < \epsilon$ and hence T has single valued extension property in λ_0 . [3, Theorem 15] asserts that $\alpha(T - \lambda_0 I) < \infty$. Then λ_0 is not in $\sigma_{ab}(T)$. It is a contradiction. It induces that $\partial\sigma_{ab}(T) \subseteq \sigma_{ea}(T)$ and hence $\eta(\sigma_{ab}(T)) = \eta(\sigma_{ea}(T))$. Similarly, for every $T_1 \in B(H)$ and $T_2 \in B(K)$, $\eta(\sigma_{ea}(T_1) \cup \sigma_{ea}(T_2)) = \eta(\sigma_{ab}(T_1) \cup \sigma_{ab}(T_2))$.

Thus for every $C \in B(K, H)$,

$$\begin{aligned}\eta(\sigma_{ab}(M_C)) &= \eta(\sigma_{ea}(M_C)) = \eta(\sigma_w(M_C)) \\ &= \eta(\sigma_w(A) \cup \sigma_w(B)) \quad [9, \text{Theorem 6}] \\ &= \eta(\sigma_{ea}(A) \cup \sigma_{ea}(B)) \\ &= \eta(\sigma_{ab}(A) \cup \sigma_{ab}(B)). \quad \square\end{aligned}$$

In the proof of Lemma 2.1, the result $\eta(\sigma_{ab}(A)) = \eta(\sigma_{ea}(A))$ is known [11, Corollary 2.11]. For a seek of completeness, we prove the fact again.

We know $\alpha(M_C) < \infty$ implies $\alpha(A) < \infty$. But if both A and B have finite ascents, then:

Lemma 2.2. *For a given pair (A, B) of operators, if both A and B have finite ascents, then for every $C \in B(K, H)$, M_C has finite ascent.*

Proof. Suppose $\alpha(A) = p$ and $\alpha(B) = q$, let $n = \max\{p, q\}$. For every $C \in B(K, H)$, if we have $N(M_C^{2n+1}) = N(M_C^{2n})$, we get the result. So we only need to prove $N(M_C^{2n+1}) \subseteq N(M_C^{2n})$.

If $u_0 \in N(M_C^{2n+1})$ and suppose $u_0 = (x_0, y_0)$. Then:

$$\begin{aligned}0 &= M_C^{2n+1}(x_0, y_0) \\ &= (A^{2n+1}x_0 + A^{2n}Cy_0 + A^{2n-1}CB y_0 + \cdots \\ &\quad + A^nCB^n y_0 + \cdots + CB^{2n}y_0, B^{2n+1}y_0),\end{aligned}$$

then $B^{2n+1}y_0 = 0$ and

$$A^{2n+1}x_0 + A^{2n}Cy_0 + A^{2n-1}CB y_0 + \cdots + A^nCB^n y_0 + \cdots + CB^{2n}y_0 = 0.$$

So $y_0 \in N(B^{2n+1}) = N(B^n)$, thus

$$A^{2n+1}x_0 + A^{2n}Cy_0 + A^{2n-1}CB y_0 + \cdots + A^{n+1}CB^{n-1}y_0 = 0,$$

that is

$$A^{n+1}[A^n x_0 + A^{n-1}Cy_0 + A^{n-2}CB y_0 + \cdots + CB^{n-1}y_0] = 0,$$

and hence

$$A^n x_0 + A^{n-1}Cy_0 + A^{n-2}CB y_0 + \cdots + CB^{n-1}y_0 \in N(A^{n+1}) = N(A^n).$$

Then

$$A^{2n}x_0 + A^{2n-1}Cy_0 + A^{2n-2}CB y_0 + \cdots + A^nCB^{n-1}y_0 = 0.$$

Now we get that

$$\begin{aligned}&(A^{2n}x_0 + A^{2n-1}Cy_0 + \cdots + A^nCB^{n-1}y_0 \\ &\quad + A^{n-1}CB^n y_0 + \cdots + CB^{2n-1}y_0, B^{2n}y_0) = 0,\end{aligned}$$

which means $M_C^{2n}u_0 = 0$ and hence $u_0 \in N(M_C^{2n})$. So $N(M_C^{2n+1}) = N(M_C^{2n})$, and hence M_C has finite ascent. \square

Let $\sigma_d(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not surjective}\}$ be the surjective spectrum of A .

Theorem 2.3. Suppose that $A \in B(H)$ and $B \in B(K)$, then for every $C \in B(K, H)$, there is equality

$$\sigma_{ab}(A) \cup \sigma_{ab}(B) = \sigma_{ab}(M_C) \cup \mathcal{G},$$

where \mathcal{G} is the union of the certain of the holes in $\sigma_{ab}(M_C)$ which happen to be subsets of $\sigma_d(A) \cap \sigma_{ab}(B)$.

Proof. Following from Lemma 2.2, for every $C \in B(K, H)$,

$$\eta(\sigma_{ab}(A) \cup \sigma_{ab}(B)) = \eta(\sigma_{ab}(M_C)). \quad (2.1)$$

First we claim that

$$\sigma_{ab}(A) \subseteq \sigma_{ab}(M_C) \subseteq [\sigma_{ab}(A) \cup \sigma_{ab}(B)]. \quad (2.2)$$

In fact, for the second inclusion, if λ_0 is not in $\sigma_{ab}(A) \cup \sigma_{ab}(B)$, then $A - \lambda_0 I$ and $B - \lambda_0 I$ are all upper semi-Fredholm operators with finite ascent. Thus $M_C - \lambda_0 I$ has finite ascent (Lemma 2.2). Since $\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}$ is invertible and

$$M_C - \lambda_0 I = \begin{pmatrix} I & 0 \\ 0 & B - \lambda_0 I \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} A - \lambda_0 I & 0 \\ 0 & I \end{pmatrix},$$

it follows that $M_C - \lambda_0 I$ is upper semi-Fredholm operator. Then λ_0 is not in $\sigma_{ab}(M_C)$. For the first inclusion, if $M_C - \lambda_0 I \in \Phi_+(H \oplus K)$ with finite ascent, then $A - \lambda_0 I \in \Phi_+(H)$. For every $n \in \mathbb{N}$, since $N[(A - \lambda_0 I)^n] \oplus \{0\} \subseteq N[(M_C - \lambda_0 I)^n]$, we know $A - \lambda_0 I$ has finite ascent. This prove (2.2).

(2.1) and (2.2) asserts that $\partial\sigma_{ab}(B) \subseteq \sigma_{ab}(M_C)$. Following we will prove that

$$[\sigma_{ab}(A) \cup \sigma_{ab}(B)] \setminus \sigma_{ab}(M_C) \subseteq \sigma_d(A) \cap \sigma_{ab}(B). \quad (2.3)$$

Let $\lambda_0 \in [\sigma_{ab}(A) \cup \sigma_{ab}(B)] \setminus \sigma_{ab}(M_C)$, then $\lambda_0 \in \sigma_{ab}(B)$ but λ_0 is not in $\sigma_{ab}(A)$. Thus there exists $\epsilon > 0$ such that $M_C - \lambda I$ and $A - \lambda I$ is bounded below [8, Lemma 2.5], $d(A - \lambda I)$ is constant and $d(A - \lambda I) \leq d(A - \lambda_0 I)$ if $0 < |\lambda - \lambda_0| < \epsilon$. There are two cases to consider.

Case 1. If there exists λ_1 such that $0 < |\lambda_1 - \lambda_0| < \epsilon$ and $R(B - \lambda_1 I)$ is not closed, then [6, Theorem 1] asserts that $d(A - \lambda_1 I) = \infty$. Then $d(A - \lambda_0 I) = \infty$, which means that $\lambda_0 \in \sigma_d(A)$;

Case 2. If there exists λ_1 such that $0 < |\lambda_1 - \lambda_0| < \epsilon$ and $R(B - \lambda_1 I)$ is closed, using [6, Theorem 1] again, $d(A - \lambda_1 I) \geq n(B - \lambda_1 I)$. If $d(A - \lambda_1 I) = 0$, then $A - \lambda_1 I$ is invertible because $A - \lambda_1 I$ is bounded below. Hence $A - \lambda I$ is invertible if $0 < |\lambda - \lambda_0| < \epsilon$. It follows that $B - \lambda I$ is bounded below if $0 < |\lambda - \lambda_0| < \epsilon$,

which means that $\lambda_0 \in \text{iso } \sigma_a(B)$. Then $\lambda_0 \in \partial\sigma_{ab}(B) \subseteq \sigma_{ab}(M_C)$. It is in contradiction to the fact that λ_0 is not in $\sigma_{ab}(M_C)$. So $d(A - \lambda_0 I) \geq d(A - \lambda_1 I) > 0$, which means $\lambda_0 \in \sigma_d(A)$.

(2.1) says that the passage from $\sigma_{ab}(M_C)$ to $\sigma_{ab}(A) \cup \sigma_{ab}(B)$ is the filling in certain of the holes in $\sigma_{ab}(M_C)$. But by (2.3), $[\sigma_{ab}(A) \cup \sigma_{ab}(B)] \setminus \sigma_{ab}(M_C)$ is contained in $\sigma_d(A) \cap \sigma_{ab}(B)$, then the filling in certain of the holes in $\sigma_{ab}(M_C)$ should occur in $\sigma_d(A) \cap \sigma_{ab}(B)$. The proof is finished. \square

Since $\text{acc } \sigma_a(B) \subseteq \sigma_{ab}(B)$, it follows that $\sigma_{ab}(B)$ has no interior points if and only if $\sigma_a(B)$ has no interior points. Using [6, Theorem 2], we have:

Corollary 2.4. *If $\sigma_d(A) \cap \sigma_{ab}(B)$ has no interior points, then for every $C \in B(K, H)$,*

$$\sigma_a(M_C) = \sigma_a(A) \cup \sigma_a(B)$$

and

$$\sigma_{ab}(M_C) = \sigma_{ab}(A) \cup \sigma_{ab}(B).$$

3. Weyl's theorem for 2×2 upper triangular operator matrices

In this section, we consider the following questions: If Weyl's (a-Weyl's) theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, when does it hold for $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$? We have:

Theorem 3.1. *If $\sigma_d(A) \cap \sigma_{ab}(B)$ ($\sigma_d(A) \cap \sigma_{SF_+}(B)$) has no interior points, then for every $C \in B(K, H)$,*

- (a) *Browder's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \implies \text{Browder's theorem holds for } \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$;*
- (b) *a-Browder's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \implies \text{a-Browder's theorem holds for } \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$.*

Proof. (a) First we will prove that for every $C \in B(K, H)$, if Browder's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, then $\sigma_w(M_C) = \sigma_b(M_C)$. We only need to prove $\sigma_b(M_C) \subseteq \sigma_w(M_C)$.

Suppose that $M_C - \lambda_0 I$ is Weyl. Then $A - \lambda_0 I \in \Phi_+(H)$ and $B - \lambda_0 I \in \Phi_-(K)$. By perturbation theory of semi-Fredholm operator, there exists $\epsilon > 0$ such that $M_C - \lambda I$ is Weyl and $N(M_C - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(M_C - \lambda I)^n]$, $n(M_C - \lambda I)$ and $d(M_C -$

λI) are all constant, $A - \lambda I \in \Phi_+(H)$, $B - \lambda I \in \Phi_-(K)$ if $0 < |\lambda - \lambda_0| < \epsilon$. And also $A - \lambda I$ is Fredholm if and only if $B - \lambda I$ is Fredholm if $|\lambda - \lambda_0| < \epsilon$.

Case 1. Suppose that λ_0 is not in $\sigma_d(A)$ or $\lambda_0 \in \partial\sigma_d(A)$.

There exists λ_1 such that $0 < |\lambda_1 - \lambda_0| < \epsilon$ and $R(A - \lambda_1 I) = H$. Then $A - \lambda_1 I$ is Fredholm and hence $B - \lambda_1 I$ is Fredholm. Since $\text{ind}(A - \lambda_1 I) + \text{ind}(B - \lambda_1 I) = \text{ind}(M_C - \lambda_1 I) = 0$, we get that $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} - \lambda_1 I$ is Weyl. Weyl's theorem

holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, then $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} - \lambda_1 I$ is Browder. Thus both $A - \lambda_1 I$ and $B - \lambda_1 I$ have finite ascents and hence $M_C - \lambda_1 I$ has finite ascent (Lemma 2.2). [12, Lemma 3.4] asserts that $N(M_C - \lambda_1 I) = N(M_C - \lambda_1 I) \cap \bigcap_{n=1}^{\infty} R[(M_C - \lambda_1 I)^n] = \{0\}$, then $M_C - \lambda_1 I$ is invertible. Since $n(M_C - \lambda I)$ and $d(M_C - \lambda I)$ are all constant if $0 < |\lambda - \lambda_0| < \epsilon$, then $M_C - \lambda I$ is invertible if $0 < |\lambda - \lambda_0| < \epsilon$, which means that $\lambda_0 \in \text{iso } \sigma(M_C)$. Since $M_C - \lambda_0 I$ is Weyl and $\lambda_0 \in \text{iso } \sigma(M_C)$, $M_C - \lambda_0 I$ must be Browder.

Case 2. Suppose $\lambda_0 \in \text{int } \sigma_d(A)$.

Since $\sigma_d(A) \cap \sigma_{ab}(B)$ has no interior points, it follows that λ_0 is not in $\sigma_{ab}(B)$ or $\lambda_0 \in \partial\sigma_{ab}(B)$. Then there exists λ_1 such that $0 < |\lambda_1 - \lambda_0| < \epsilon$ and $B - \lambda_1 I \in \Phi_+(K)$ with finite ascent. Then $B - \lambda_1 I$ is Fredholm and hence $A - \lambda_1 I$ is Fredholm. The following proof is same to the proof in Case 1.

From the above proof, we claim that Browder's theorem holds for M_C for every $C \in B(K, H)$.

(b) Following we will prove for every $C \in B(K, H)$, if a-Browder's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, then $\sigma_{ea}(M_C) = \sigma_{ab}(M_C)$. We only need to prove that $\sigma_{ab}(M_C) \subseteq \sigma_{ea}(M_C)$.

If λ_0 is not in $\sigma_{ea}(M_C)$, then $M_C - \lambda_0 \in \Phi_+^-(H \oplus K)$ and hence $A - \lambda_0 I \in \Phi_+(H)$. By perturbation theory of semi-Fredholm operator, there exists $\epsilon > 0$ such that $M_C - \lambda I \in \Phi_+^-(H \oplus K)$, $N(M_C - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(M_C - \lambda I)^n]$, $A - \lambda I \in \Phi_+(H)$ and both $n(M_C - \lambda I)$ and $d(A - \lambda I)$ are constant if $0 < |\lambda - \lambda_0| < \epsilon$. Same to the case of Browder's theorem, we have two cases to consider:

Case 1. If λ_0 is not in $\sigma_d(A)$ or $\lambda_0 \in \partial\sigma_d(A)$, then there exists λ_1 such that $0 < |\lambda_1 - \lambda_0| < \epsilon$ and $R(A - \lambda_1 I) = H$. Thus $R(A - \lambda I) = H$ for all $0 < |\lambda - \lambda_0| < \epsilon$.

Following, we will prove that $B - \lambda I \in \Phi_+(K)$ if $0 < |\lambda - \lambda_0| < \epsilon$. First we will prove that $n(B - \lambda I) < \infty$. If not, there exists λ_1 such that $0 < |\lambda_1 - \lambda_0| < \epsilon$ and $n(B - \lambda_1 I) = \infty$. Suppose $\{y_n\}_{n=1}^{\infty}$ is an orthonormal sequence in $N(B - \lambda_1 I)$. Since $A - \lambda_1 I$ is surjective, there exists a sequence $\{x_n\}$ in H such that

$$(A - \lambda_1 I)x_n = Cy_n \quad \text{for each } n = 1, 2, \dots$$

But then

$$\begin{pmatrix} A - \lambda_1 I & C \\ 0 & B - \lambda_1 I \end{pmatrix} (x_n, -y_n) = (0, 0) \quad \text{for each } n = 1, 2, \dots,$$

which means that $N(M_C - \lambda_1 I)$ is infinite dimensional, a contradiction. Second, we will prove that $R(B - \lambda I)$ is closed if $0 < |\lambda - \lambda_0| < \epsilon$. Suppose that $(B - \lambda I)y_n \rightarrow y_1 (n \rightarrow \infty)$. Then $\begin{pmatrix} I & 0 \\ 0 & B - \lambda I \end{pmatrix} (0, y_n) \rightarrow (0, y_1)$. Since $\begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} A - \lambda I & 0 \\ 0 & I \end{pmatrix}$ is surjective, there exists $(u_n, v_n) \in H \oplus K$ such that $\begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} A - \lambda I & 0 \\ 0 & I \end{pmatrix} (u_n, v_n) = (0, y_n)$. Then

$$\begin{aligned} (M_C - \lambda I)(u_n, v_n) &= \begin{pmatrix} I & 0 \\ 0 & B - \lambda I \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} A - \lambda I & 0 \\ 0 & I \end{pmatrix} (u_n, v_n) \\ &= \begin{pmatrix} I & 0 \\ 0 & B - \lambda I \end{pmatrix} (0, y_n) \rightarrow (0, y_1). \end{aligned}$$

Since $R(M_C - \lambda I)$ is closed, there exists $(x_0, y_0) \in H \oplus K$ such that $(M_C - \lambda I)(x_0, y_0) = (0, y_1)$. Then $(B - \lambda_0 I)y_0 = y_1$ and hence $R(B - \lambda I)$ is closed if $0 < |\lambda - \lambda_0| < \epsilon$.

Now we have that $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} - \lambda I \in \Phi_+(H \oplus K)$ and $\text{ind}\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} - \lambda I\right) = \text{ind}(M_C - \lambda I) \leq 0$. Since a-Browder's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, we know $\alpha\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} - \lambda I\right) < \infty$ and hence $\alpha(M_C - \lambda I) < \infty$ (Lemma 2.2). Then $N(M_C - \lambda I) = N(M_C - \lambda I) \cap \bigcap_{n=1}^{\infty} R[(M_C - \lambda I)^n] = \{0\}$, which means that $M_C - \lambda I$ is bounded below and hence $\lambda_0 \in \text{iso } \sigma_a(M_C)$. So M_C has single valued extension property in λ_0 . [3, Theorem 15] tells us that $M_C - \lambda_0 I \in \Phi_+(H \oplus K)$ with finite ascent, that is λ_0 is not in $\sigma_{ab}(M_C)$.

Case 2. If $\lambda_0 \in \text{int } \sigma_a(A)$, then λ_0 is not in $\sigma_{ab}(B)$ or $\lambda_0 \in \partial \sigma_{ab}(B)$. Then there exists λ_1 such that $0 < |\lambda_1 - \lambda_0| < \epsilon$ and $B - \lambda_1 I \in \Phi_+(K)$. Thus $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} - \lambda_1 I \in \Phi_+^-(H \oplus K)$. Since a-Browder's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, both $A - \lambda_1 I$ and $B - \lambda_1 I$ have finite ascents and hence $M_C - \lambda_1 I$ has finite ascent. Same to the proof of Case 1, $M_C - \lambda_1 I$ is bounded below. Since $n(M_C - \lambda I)$ is constant if $0 < |\lambda - \lambda_0| < \epsilon$, it follows that $M_C - \lambda I$ is bounded below if $0 < |\lambda - \lambda_0| < \epsilon$. Then $\lambda_0 \in \text{iso } \sigma_a(M_C)$ and hence λ_0 is not in $\sigma_{ab}(M_C)$. The proof is completed. \square

Remark 3.2. Theorem 3.1 may fail for “a-Weyl’s theorem” even with the additional assumption that “a-Weyl’s theorem holds for A and B and both A and B are all a-isoloid”. For example, let $A, B, C \in B(\ell_2)$ are defined by

$$\begin{aligned} A(x_1, x_2, x_3, \dots) &= (0, x_1, 0, x_2, 0, x_3, \dots); \\ B(x_1, x_2, x_3, \dots) &= (0, x_2, 0, x_4, 0, x_6, \dots); \\ C(x_1, x_2, x_3, \dots) &= (0, 0, 0, 0, \frac{1}{3}x_3, 0, \frac{1}{5}x_5, \dots). \end{aligned}$$

Then

$$\begin{aligned}\sigma_a(A) &= \sigma_{ea}(A) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}, \\ \pi_{00}^a(A) &= \emptyset, \sigma_d(A) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\} \quad \text{and} \\ \sigma_a(B) &= \sigma_{ea}(B) = \sigma_{ab}(B) = \{0, 1\} \quad \text{and} \quad \pi_{00}^a(B) = \emptyset,\end{aligned}$$

which says that both A and B are a -isoloid and a -Weyl's theorem holds for A and B . Clearly, $\sigma_d(A) \cap \sigma_{ab}(B)$ has no interior points. Also a straightforward calculation shows that

$$\begin{aligned}\sigma_a \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} &= \sigma_{ea} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \sigma_a \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \sigma_{ea} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \\ &= \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{0\} \quad \text{and} \\ \pi_{00}^a \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} &= \emptyset, \quad \pi_{00}^a \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \{0\},\end{aligned}$$

which implies that a -Weyl's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, but a -Weyl's theorem fails for $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$.

But we have the following results:

Theorem 3.3. *Suppose that $\sigma_d(A)$ has no interior points. If A is a -isoloid and a -Weyl's theorem holds for A , then for every $B \in B(K)$ and $C \in B(K, H)$,*

$$\begin{aligned}&a\text{-Weyl's theorem holds for } \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \\ \implies &a\text{-Weyl's theorem holds for } \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.\end{aligned}$$

Proof. From Theorem 3.1, $\sigma_a(M_C) \setminus \sigma_{ea}(M_C) \subseteq \pi_{00}^a(M_C)$.

Conversely, suppose that $\lambda_0 \in \pi_{00}^a(M_C)$. Then $M_C - \lambda I$ is bounded below if $|\lambda - \lambda_0|$ is sufficiently small and hence λ is not in $\sigma_a(M_C)$. Since $\sigma_d(A)$ has no interior points, by Corollary 2.4, $\sigma_a(M_C) = \sigma_a(A) \cup \sigma_a(B) = \sigma_a \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Then λ is not in $\sigma_a \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ if $|\lambda - \lambda_0|$ is sufficiently small, that is $\lambda_0 \in \text{iso } \sigma_a \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Without loss of generality, we suppose that $\lambda_0 \in \sigma_a(A)$, then $\lambda_0 \in \text{iso } \sigma_a(A)$. Since $N(A - \lambda_0 I) \oplus \{0\} \subseteq N(M_C - \lambda_0 I)$, we know that $n(A - \lambda_0 I) < \infty$. A is a -isoloid, then $\lambda_0 \in \pi_{00}^a(A)$. Since a -Weyl's theorem holds for A , it follows that $A - \lambda_0 I \in \Phi_+(H)$ and $\alpha(A - \lambda_0 I) < \infty$. The condition $\sigma_d(A)$ has no interior points asserts that λ_0

is not in $\sigma_d(A)$ or $\lambda_0 \in \partial\sigma_d(A)$. Then in any neighborhood U of λ_0 , there exists $\lambda_1 \in U$ such that $R(A - \lambda_1 I) = H$. By perturbation theory of upper semi-Fredholm operator $A - \lambda_0 I$, we get that $A - \lambda I$ is invertible and $\text{ind}(A - \lambda_0 I) = \text{ind}(A - \lambda I) = 0$ if $|\lambda - \lambda_0|$ is sufficiently small, which means that $A - \lambda_0 I$ is Weyl with finite ascent. [12, Theorem 4.5] asserts that $A - \lambda_0 I$ is Browder. Using the same way in Theorem 2.4 in [10], we get that $0 < \dim[N(A - \lambda_0 I) \oplus N(B - \lambda_0 I)] < \infty$, which implies that $\lambda_0 \in \pi_{00}^a \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Since a-Weyl's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, it follows that $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} - \lambda_0 I \in \Phi_+^-(H \oplus K)$. Hence $M_C - \lambda_0 I \in \Phi_+^-(H \oplus K)$, then $\lambda_0 \in \sigma_a(M_C) \setminus \sigma_{ea}(M_C)$. Now we have proved that $\sigma_a(M_C) \setminus \sigma_{ea}(M_C) = \pi_{00}^a(M_C)$, which means that a-Weyl's theorem holds for M_C for every $C \in B(K, H)$. \square

In Remark 3.2, we know that Theorem 3.1 may fail for “a-Weyl's theorem” even with the additional assumption that A is a-isoloid and a-Weyl's theorem holds for A . But for Weyl's theorem, we have:

Theorem 3.4. Suppose that $\sigma_d(A) \cap \sigma_{ab}(B)$ has no interior points. If A is isoloid and Weyl's theorem holds for A , then for every $C \in B(K, H)$,

$$\begin{aligned} & \text{Weyl's theorem holds for } \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \\ \implies & \text{Weyl's theorem holds for } \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}. \end{aligned}$$

Proof. Theorem 3.1 tells us that for every $C \in B(K, H)$, $\sigma(M_C) \setminus \sigma_w(M_C) \subseteq \pi_{00}(M_C)$. Conversely, let $\lambda_0 \in \pi_{00}(M_C)$, then $M_C - \lambda I$ is invertible if $|\lambda - \lambda_0|$ is sufficiently small. Thus $A - \lambda I$ is bounded below and $B - \lambda I$ is surjective if $|\lambda - \lambda_0|$ is sufficiently small. Since $\sigma_d(A) \cap \sigma_{ab}(B)$ has no interior points, it follows that $\sigma_a(M_C) = \sigma_a(A) \cup \sigma_a(B)$. Then λ is not in $\sigma_a(B)$ if $|\lambda - \lambda_0|$ is sufficiently small. Thus $B - \lambda I$ is invertible and hence $A - \lambda I$ is invertible if $|\lambda - \lambda_0|$ is sufficiently small. Now we have proved that $\lambda_0 \in \text{iso } \sigma \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, the following proof is same to the proof in Theorem 2.4 in [10]. \square

Remark 3.5. Theorem 3.4 in this paper is not compatible with Theorem 2.4 in [10]. For example:

(a) Suppose that $A, B \in B(\ell_2)$ are defined by

$$\begin{aligned} A(x_1, x_2, x_3, \dots) &= (x_2, x_4, x_6, \dots); \\ B(x_1, x_2, x_3, \dots) &= (0, x_1, 0, x_2, 0, x_3, 0, \dots). \end{aligned}$$

Then

$$\begin{aligned}\sigma(A) &= \sigma_w(A) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}, \quad \pi_{00}(A) = \emptyset, \\ \sigma_d(A) &= \{\lambda \in \mathbb{C} : |\lambda| = 1\}; \\ \sigma_{SF_-}(A) &= \{\lambda \in \mathbb{C} : |\lambda| = 1\}, \quad \sigma_e(A) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}; \\ \sigma(B) &= \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}, \quad \sigma_{ab}(B) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}; \\ \sigma_{SF_+}(B) &= \{\lambda \in \mathbb{C} : |\lambda| = 1\}, \quad \sigma_e(B) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}.\end{aligned}$$

We have:

- (i) $\sigma_d(A) \cap \sigma_{ab}(B)$ has no interior points;
- (ii) A is isoloid and Weyl's theorem holds for A ;
- (iii) Both $SP(A)$ and $SP(B)$ have pseudoholes;
- (iv) $\sigma \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \sigma_w \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and $\pi_{00} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \emptyset$. Then Weyl's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

Then using Theorem 3.4 in this note, for every $C \in B(K, H)$, Weyl's theorem holds for $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. But using Theorem 2.4 in [10], we do not know whether Weyl's theorem holds for M_C for every $C \in B(K, H)$.

(b) Suppose $A, B \in B(\ell_2)$ are defined by

$$\begin{aligned}A(x_1, x_2, x_3, \dots) &= (0, x_1, x_2, x_3, \dots); \\ B(x_1, x_2, x_3, \dots) &= (x_2, x_4, x_6, \dots).\end{aligned}$$

Then

$$\begin{aligned}\sigma(A) &= \sigma_w(A) = \sigma_d(A) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}, \quad \pi_{00}(A) = \emptyset; \\ \sigma_e(A) &= \sigma_{SF_+}(A) = \sigma_{SF_-}(A); \\ \sigma(B) &= \sigma_w(B) = \sigma_{ab}(B) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}.\end{aligned}$$

We get that:

- (i) $\sigma_d(A) \cap \sigma_{ab}(B)$ has interior points;
- (ii) A is isoloid and Weyl's theorem holds for A ;
- (iii) $SP(A)$ has no pseudoholes;
- (iv) $\sigma \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \sigma_w \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $\pi_{00} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \emptyset$, then Weyl's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

Using Theorem 2.4 in [10], for every $C \in B(K, H)$, Weyl's theorem holds for $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. But using Theorem 3.4 in this note, we do not know whether the result is true.

Similar to the proof in Theorem 3.3, we can prove that:

Theorem 3.6. *Suppose that $\sigma_d(A) \cap \sigma_{ab}(B)$ has no interior points. If $SP(A)$ has no pseudoholes (or $\sigma_e(A) = \sigma_{ab}(A)$) and if A is an a -isoloid operator for which a -Weyl's theorem holds, then for every $C \in B(K, H)$,*

$$\begin{aligned} & a\text{-Weyl's theorem holds for } \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \\ \implies & a\text{-Weyl's theorem holds for } \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}. \end{aligned}$$

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